THE STRATEGY APPROVAL DECISION:
A SHARPE RATIO INDIFFERENCE CURVE APPROACH

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ABSTRACT

The problem of capital allocation to a set of strategies could be partially avoided or at least greatly simplified with an appropriate strategy approval decision process. This paper proposes such procedure. We begin by splitting the capital allocation problem into two tasks: *Strategy approval* and *portfolio optimization*. Then we argue that the goal of the second task is to beat a naïve benchmark, and the goal of the first task is to identify which strategies improve the performance of such naïve benchmark. This is a sensible approach, as it doesn't leave all the work to the optimizer, thus adding robustness to the final outcome.

We introduce the concept of Sharpe ratio Indifference Curve, which represents the space of pairs (candidate strategy's Sharpe ratio, candidate strategy's correlation to the approved set) for which the Sharpe ratio of the expanded approved set remains constant. This proves that selecting strategies (or portfolio managers) solely based on past Sharpe ratio will lead to suboptimal outcomes, particularly when we ignore the impact that these decisions will have on the average correlation of the portfolio. Finally, we show that these results have important practical business implications with respect to the way hedge funds hire, fire and structure payouts.[1]

Keywords: Portfolio theory, Sharpe ratio, pairwise correlation, indifference curve, diversification, free call option.

JEL classifications: C02, G11, G14, D53.

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1.- INTRODUCTION

The problem of allocating capital to Portfolio Managers (PMs) or strategies is typically addressed using a variation of Markowitz's (1952) approach. This method is agnostic as to the criterion used to pre-select those PMs. In this paper we will show that the standard criterion used by the hedge fund industry to hire and fire PMs may indeed lead to suboptimal capital allocations.

In a series of papers, Sharpe (1966, 1975, 1994) introduced a risk-adjusted measure of investment's performance. This measure, universally known as the Sharpe ratio (SR), has become the gold-standard to evaluate PMs in the hedge fund industry. Most hedge funds require any candidate manager or strategy to pass a set of fixed thresholds with regards to SR (De Souza and Gokcan 2004) and track record length (Bailey and López de Prado 2012) in order to be allocated capital. However, ignoring the candidate strategy's correlation to the set of approved strategies is counterintuitive. A strategy with lower SR may introduce more diversification than other strategy with higher SR but also higher correlation to the approved set (more on that in Section 3).

Before considering a candidate strategy for approval, it is critical to determine not only its expected SR, but also its average correlation against the approved set of strategies. We demonstrate that *there is no fixed SR threshold which we should demand for strategy approval*. We must jointly look at the candidate's SR *and* how it fits in the existing menu of strategies. We will often find situations in which a highly performing candidate strategy should be declined due to its high average correlation with the existing set. Conversely, a low performing candidate strategy may be approved because its diversification potential offsets the negative impact on the average SR.

Looking at the combined effect that a candidate's SR and correlation will have on the approved set also addresses a fundamental critique to the "fixed SR threshold" approach currently applied by most hedge funds. Such fixed threshold tends to favor higher over lower frequency strategies. But considering the (low) correlation that the latter strategies have with respect to the former, lower frequency strategies will have a fairer chance of being approved under the new approach hereby presented.

Finally, we explain how this new strategy approval decision process could lead to new business arrangements in the hedge fund industry. Emulating the performance of "star-PMs" through a large number of uncorrelated low-SR PMs will give the opportunity for hedge funds to internalize features that cannot be appropriated by the individual PMs.

The rest of this paper is structured as follows: Section 2 presents a few propositions on the naïve benchmark's performance. Section 3 introduces the *SR Indifference Curve*. Section 4 makes a specific proposal for the process of approving strategies. Section 5 proposes a new business arrangement based of this strategy approval process. Section 6 lists the conclusions. The appendices present the mathematical proofs to these propositions.

2.- PROPOSITIONS

The following propositions differ from standard portfolio theory in a number of ways:

1. They discuss the allocation of capital across strategies or PMs, rather than assets.
2. They are based on the assumption of an Equal Volatility Weightings (or *naïve*) benchmark (DeMiguel, Garlappi and Uppal 2009).
3. This benchmark allows us to split the capital allocation problem into two sub-problems:
    a. **Strategy Approval**: The process by which a candidate strategy is approved to be part of a portfolio.
    b. **Portfolio Optimization**: The process that determines the optimal amount of capital to be allocated to each strategy within a portfolio.
4. Key principles of the approach discussed here are:
    a. The goal of the *Portfolio Optimization* process is to beat the performance of a naïve benchmark.
    b. The goal of the *Strategy Approval* process is to set a benchmark as high as possible (ideally, to the point that no portfolio optimization is required at all!).

Capital allocation to PMs or strategies is typically done without consideration of the strategy approval process (Lhabitant 2004). This means that the portfolio optimization step may have to deal with PMs or strategies pre-selected according to a criterion that will lead to suboptimal capital allocations. This paper is dedicated to show how to achieve the goal of the strategy approval (or hiring) process by setting a naïve benchmark that the portfolio optimization step must beat. In doing so, we avoid leaving the entire decision to techniques that have been criticized for their lack of robustness (Best and Brauer 1991).

2.1.- BENCHMARK PORTFOLIO

2.1.1.- STATEMENT

The performance of an 'Equal Volatility Weights' benchmark (SR_B) is fully characterized in terms of:

1. Number of approved strategies (S).
2. Average SR among strategies ($\overline{SR}$).
3. Average off-diagonal correlations among strategies ($\bar{\rho}$).

In particular, adding strategies (S) with the same $\overline{SR}$ and $\bar{\rho}$ does improve SR_B.

Sections A.1 and A.2. prove this statement.

2.1.2.- EXAMPLE

Following Eq. (6) in Appendix 2, it will take 16 strategies with $\overline{SR} = 0.75$ and $\bar{\rho} = 0.2$ to obtain a benchmark with SR of 1.5. Should the average individual risk-adjusted performance decay to $\overline{SR} = 0.5$, the benchmark's SR will drop to 1. Exhibit 1 illustrates the point that, if on top of that performance degradation the individual pairwise correlation raises to $\bar{\rho} = 0.3$, the benchmark's SR will be only 0.85.

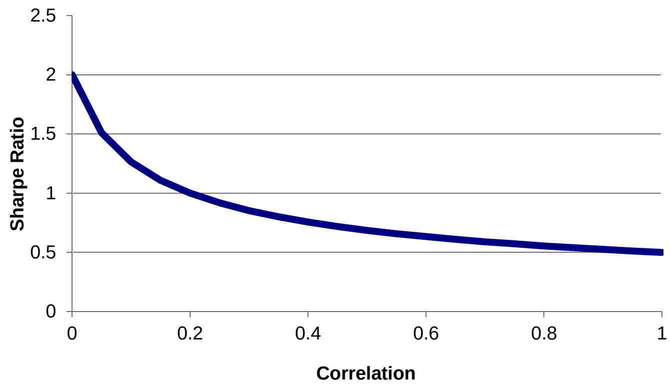

Exhibit 1 – Benchmark's SR for S=16 strategies with $\overline{SR} = 0.5$
as a function of the average correlation

2.1.3.- PRACTICAL IMPLICATIONS

This proposition allows us to estimate the benchmark's SR without requiring knowledge of the individual strategy's SRs or their pairwise correlations. Average volatility is not a necessary input. All is needed is S, $\overline{SR}$, and $\bar{\rho}$. This makes possible the simulation of performance degradation or correlation stress-test scenarios, as illustrated in the previous epigraph.

2.2.- ON PERFORMANCE DEGRADATION

2.2.1.- STATEMENT

The benchmark SR is a *linear* function of the average SR of the individual strategies, and a decreasing convex function of the number of strategies and the average pairwise correlation. This means that, as the number of strategies (S) increases, favoring low $\bar{\rho}$ offers a convex payoff (similar to an option) which $\overline{SR}$ does not. In the presence of performance degradation, low correlated strategies may be preferable to (supposedly) highly performing ones.

Section A.3 proves this statement.

2.2.2.- EXAMPLE

For $\overline{SR} = 0.75$, Exhibit 2 shows the benchmark's SR as a function of $\overline{SR}$ and $\bar{\rho}$ for 5 and 25 strategies (see Eqs. (7)-(8) in Appendix 3).

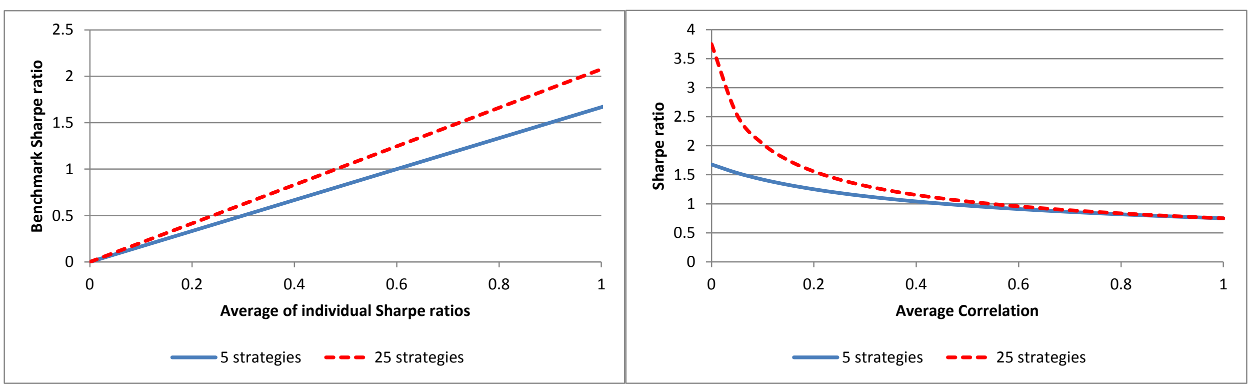

Exhibit 2 – The linear impact of $\overline{SR}$ *degradation vs. the convex payoff of* $\bar{\rho}$

2.2.3.- PRACTICAL IMPLICATIONS

This is a critical result. It implies that we may prefer low correlated strategies, even if underperforming, to overperforming but highly correlated strategies. The exact trade-off between these two characteristics will become clearer in Section 3.

2.3.- ON THE MAXIMUM ACHIEVABLE BENCHMARK SR

2.3.1.- STATEMENT

There is a limit to how much the benchmark SR can be improved by adding strategies. In particular, that limit is fully determined by:

1. Average SR among strategies ($\overline{SR}$).
2. Average off-diagonal correlations among strategies ($\bar{\rho}$).

Section A.4 proves this statement.

2.3.2.- EXAMPLE

Suppose that $\overline{SR} = 0.75$ and $\bar{\rho} = 0.2$. Regardless of how many equivalent strategies are added, the benchmark's SR will not exceed 1.68 (Exhibit 3). Higher SRs could still be obtained with a skillful (non-naïve) portfolio optimization process, but are beyond the benchmark's reach (see Eqs. (10)-(11) in Appendix 4).

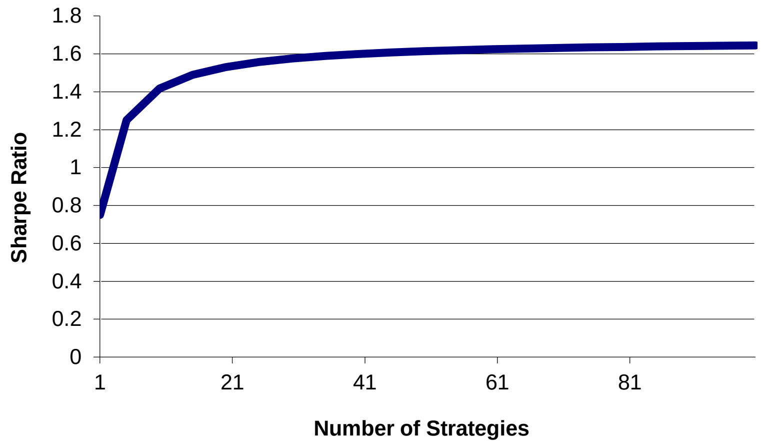

Exhibit 3 – SR of a portfolio of approved strategies with $\overline{SR} = 0.75$ *and* $\bar{\rho} = 0.2$.

2.3.3.- PRACTICAL IMPLICATIONS

In the absence of $\overline{SR}$ degradation, it would make little sense increasing the number of strategies (S) beyond a certain number. But since $\overline{SR}$ degradation is expected, there is a permanent need for building an inventory of *replacement strategies* (to offset for those decommissioned due to performance degradation or approval error (false positive)).This is consistent with Proposition 2, which offered a theoretical justification for researching as many (low correlated) strategies as possible (the convex payoff due to correlation).

2.4.- ON THE IMPACT OF A CANDIDATE STRATEGY ON THE BENCHMARK'S SR

2.4.1.- STATEMENT

A strategy being considered for approval would have an impact on the benchmark's SR (and thus its naïve targeted performance) that exclusively depends on:

1. Number of approved strategies (S).
2. Average SR among strategies ($\overline{SR}$).
3. Average off-diagonal correlations among strategies ($\bar{\rho}$).
4. Average correlation of the candidate strategies against the approved set ($\bar{\rho}_{S+1}$).
5. The candidate strategy's SR (SR_{S+1}).

Section A.5 proves this statement.

2.4.2.- EXAMPLE

Suppose the case that S=2, $\overline{SR} = 1$, $\bar{\rho} = 0.1$, thus $SR_B = 1.35$. Consider a third strategy with $SR_3 = 1$ and $\bar{\rho}_3 = 0.1$. Then, applying Eq. (12) in Appendix 5, $SR_B = 1.58$. Adding the third strategy positively impacted the benchmark's SR, even though there was no improvement on $\overline{SR} = 1$, $\bar{\rho} = 0.1$. We knew this from Proposition 1.

Let's turn now to the case where $SR_3 = 0.7$ and $\bar{\rho}_3 = 0.1$. Then, $SR_B = 1.42$. If however $\bar{\rho}_3 = 0.2$, then $SR_B = 1.32$. We were able to make those calculations without requiring additional knowledge regarding the strategy's risk or pairwise correlations.

2.4.3.- PRACTICAL IMPLICATIONS

Proposition 4 shows that $\bar{\rho}_{S+1}$ and SR_{S+1} suffice to determine the new benchmark's SR. In particular, we do not need to know each pairwise correlation, individual SRs or strategies' volatilities, which greatly simplifies simulation exercises.

3.- THE SR INDIFFERENCE CURVE (STRATEGY APPROVAL THEOREM)

The previous propositions converge into the following fundamental result.

3.1.- STATEMENT

There exists a trade-off such that we would be willing to accept a strategy with below average SR if its average correlation to the approved set is below a certain level. This determines and indifference curve as a function of:

1. Number of approved strategies (S).
2. Average SR ($\overline{SR}$).
3. Average off-diagonal correlations ($\bar{\rho}$).
4. The candidate strategy's SR (SR_{S+1}).

Section A.6 proves this statement.

3.2.- EXAMPLE

Suppose the same case as in Proposition 4, namely that $S = 2$, $\overline{SR} = 1$, $\bar{\rho} = 0.1$, thus $SR_B = 1.35$. A third strategy with $SR_3 = 1$ and $\bar{\rho}_3 = 0.1$ would lead to $SR_B = 1.58$. The theorem says that, should an alternative third strategy deliver $SR_3 = 1.5$ instead, we would be indifferent for $\bar{\rho}_3 = 0.425$ (see Eq. (13) in Appendix 6). Beyond that correlation threshold, the alternative with *higher* SR should be *declined*. Exhibit 4 shows the entire indifference curve.

More interestingly, we would also be indifferent to a second alternative whereby $SR_3 = -0.1$ and $\bar{\rho}_3 = -0.439$. But why would we ever approve a strategy that very likely will not make any money? Why would a hedge fund hire a PM that loses money? This probably sounds counter-intuitive, but that's where the previous math becomes helpful. The reason is, we are investing in 3 strategies. Overall, we will still have a quite positive return. True that this overall return would be *slightly* greater without the third strategy, however without it the standard deviation would also be *much* larger. All things considered, if that third strategy delivers an average correlation below -0.439, it improves the overall SR beyond $SR_B = 1.58$. In this particular example, the third strategy would behave like a call option at a premium equivalent to the $\sigma_3 SR_3$ it costs to "buy" it. Naturally, strategies with a -0.439 average correlation are hard to find, but if they presented themselves, we should consider them.

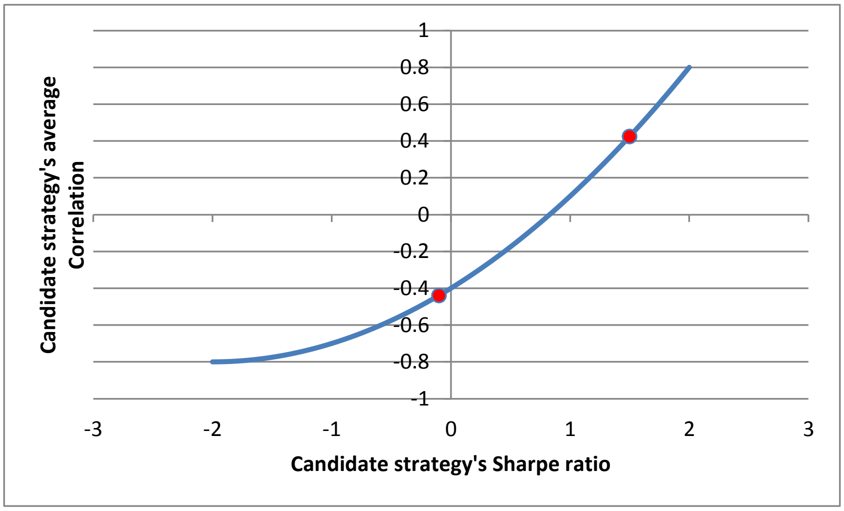

Exhibit 4 – Indifference curve between a candidate strategy's SR and its average correlation to the approved set (examples marked with red dots)

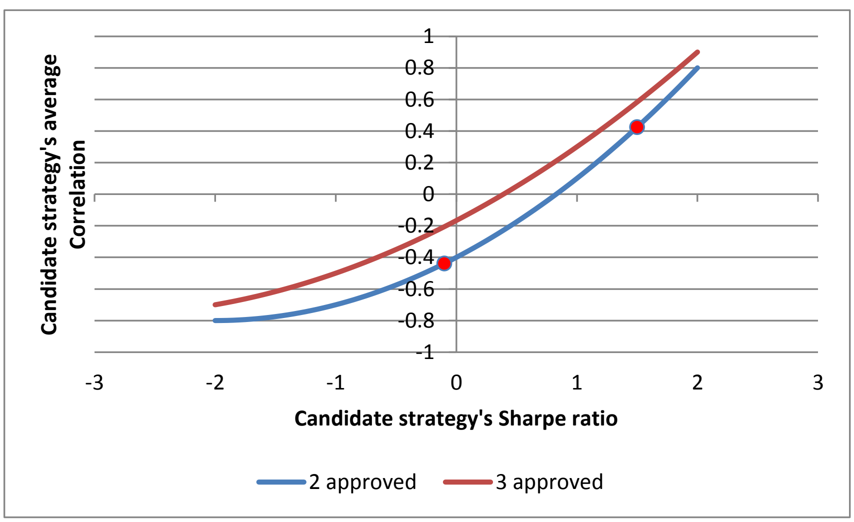

Exhibit 5 – The indifference curve is not static, and as more risk is pooled, some of the previously rejected strategies become acceptable

Finally, suppose that the first of the three alternatives is added ($SR_3 = 1$, $\bar{\rho}_3 = 0.1$). This will in turn shift the indifference curve to the left and up (see Exhibit 5). It means that pairs of $(SR_{S+1}, \bar{\rho}_{S+1})$ that fall between the red and the blue curve have now become acceptable. As the set of approved strategies pools more risk, it is able to clear room for previously rejected strategies without reducing the benchmark's overall SR.

3.3.- PRACTICAL IMPLICATIONS

For every candidate strategy, there exists an infinite group of alternative theoretical candidates whereby all deliver the same benchmark SR. The indifference curve represents the exact trade-off between a candidate strategy's SR and its average correlation against the approved set such that the benchmark's SR is preserved.

This theorem addresses a problem faced by most hedge funds: The "fixed SR threshold" strategy selection criterion represents a considerable hurdle for the lower frequency strategies. These strategies tend to have a lower annualized SR, but they also bring lower average correlations, with an overall improvement in diversification. This approach finds the balance between both components, allowing for low frequency strategies to be accepted in equal conditions as higher frequency strategies.

4.- PROPOSAL FOR A COHERENT STRATEGY APPROVAL PROCESS

Before considering a candidate strategy for approval, it is critical to determine no only its expected SR, but also its average correlation against the approved set of strategies. The above results imply that *there is no fixed SR threshold which we should demand for strategy approval*. We must jointly look at the candidate's SR *and* how it fits in the existing menu of strategies by considering:

1. Number of approved strategies (S).
2. Average SR among strategies ($\overline{SR}$).
3. Average off-diagonal correlations among strategies ($\bar{\rho}$).
4. The candidate strategy's SR (SR_{S+1}).
5. Average correlation of the candidate strategies against the approved set ($\bar{\rho}_{S+1}$).

A realistic backtest would reflect transaction costs and market impact when estimating SR_{S+1}, thus incorporating a capacity penalty in this analysis. With these inputs we can then compute SR_B (without the candidate strategy), SR_B^* (including the candidate strategy), and given SR_{S+1} for what $\bar{\rho}_{S+1}$ it occurs that $SR_B = SR_B^*$ (indifference point).

We will often find situations in which a highly performing candidate strategy should be declined due to its high average correlation with the existing set. Conversely, a low performing candidate strategy may be approved because its diversification potential offsets the negative impact on the average SR.

It is important to note that the input variables do not need to be restricted to historical estimates, but can reflect forward looking scenarios. This makes it easy to reset approval thresholds under alternative assumptions on capacity, future correlation, performance degradation, etc.

5.- BUSINESS IMPLICATIONS

Far from being a theoretical result, these propositions have a number of very practical business implications. Funds typically pay PMs a percentage of the net profits generated by their strategies. PMs do not share a percentage of the losses, which gives them exposure to the upside only. Funds are therefore writing what is called a *“free call option”* (Gregoriou et al. 2011, Lhabitant 2004). The true value of the option is proportional to the risks associated with a PM’s strategy. The better the PM’s strategy, the lower the probability of losses, therefore the cheaper the option. Conversely, the option offered to an unskilled PM is extremely expensive. For this reason, funds do not evaluate a PM’s performance in terms of average annual return, as that would not take into account the risks involved and would lead to offering the option to the wrong PMs.

The core argument presented in this paper –that SR is a bad index of whom a fund should hire or fire– seems at odds with standard business practices. The *SR Indifference Curve* shows that even PMs with a negative individual SR should be hired if they contribute enough diversification. Why is that not the case? Because of a *netting* problem: The typical business agreement is that PMs are entitled to a percentage of their individual performance, not a percentage of the fund’s performance. Even though legal clauses would prevent the fund from paying a profitable PM if the overall fund has lost money, that PM is unlikely to remain at the firm after a number of such events. This is a very unsatisfactory situation, for a number of reasons: First, funds are giving up the extra-performance predicted by the *SR Indifference Curve*. Second, funds are compelled to hire ‘star-PMs’, who will require a high portion of the performance fee. Third, funds are always under threat of losing to competitors their ‘star-PMs’, who may leave the firm with their trade secrets for a slightly better deal. In some hedge funds, PMs’ turnover is extremely high, with an average tenure of only a few years.

A way to avoid this suboptimal outcome is to offer a business deal that pays the PM a percentage of the fund’s overall performance. This would again create some tensions, as some ‘star-PMs’ could do better with their individual deals. However, Section 2.3 tells us that we can emulate the performance of a ‘star-PM’ by hiring a sufficient number of ‘average-PMs’ with low correlation to the fund’s performance. A first advantage of doing so is that ‘average-PMs’ have no bargaining power, thus we can pay them a lower proportion of the performance fee. A second advantage is that, because of the relatively low SR, they are unlikely to be poached. A third advantage is that, if we hire ‘average-PMs’ which performance have low correlation to the fund’s, we can internalize a private value to which the PMs has no access. The average-PM’s performance may exhibit a low correlation to a limited number of funds’, but not to all. In other words, the fund can capture the extra-performance postulated by the *SR Indifference Curve* without having to pay for it.

A future can therefore be envisioned in which hedge funds are set up with the following features:

- Payout is arranged in terms of funds’ overall performance, which may be superior to ‘star-driven’ funds.
- Hiring targets PMs with low SRs (even below zero, if their correlation is sufficiently negative), and therefore cheaper to find, keep and replace.

- There is a very low turnover of PMs, as they cannot take the ‘low correlation to the fund’ with them, and the low SR does not get them an individual deal.

This kind of business arrangement is particularly suitable to algorithmic hedge funds, because the prerequisite of ‘sufficient number of average-PMs’ can be easily fulfilled with average-performing trading systems. Since the SR required to put each system in production will be relatively low, they can be developed in large numbers. As long as each quant developer is involved in a limited number of those systems, their bargaining power will still be limited.

6.- CONCLUSIONS

Ideally, if a hedge fund counted with virtually uncorrelated strategies, no optimization would be required at all. Although an unrealistic scenario, it is nonetheless true that many of the problems associated with portfolio optimization could be avoided, to a great extent, with a proper procedure of strategy approval. The procedure discussed in this paper is a first step in that direction.

We have divided the capital allocation problem in two sequential phases: Strategy approval and portfolio optimization. The goal of the strategy approval phase is to rise the naïve benchmark’s performance, reducing the burden typically placed on the portfolio optimization phase. We have demonstrated that there is no fixed SR threshold which we should demand for strategy approval. Instead, there is an indifference curve of pairs (candidate strategy’s SR, candidate’s correlation to approved set) that keep the benchmark’s SR constant. At the extreme, it may be preferable to approve a candidate’s strategy with negative Sharpe if its correlation to the approved set is sufficiently negative.

These results are particularly relevant in the context of performance degradation, as they demonstrate that selecting strategies (or PMs) solely based on past SR may lead to suboptimal results, especially when we ignore the impact that these decisions will have on the average correlation of the portfolio. The practical implication is that hedge funds could emulate the performance of “star-PMs” through uncorrelated low-SR PMs, who will not have individual bargaining power.

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APPENDICES

A.1.- DEFINITIONS AND STANDING HYPOTHESIS

Suppose a collection of *S* strategies, each with Normally distributed excess returns

$$r_s \sim N(\mu_s, \sigma_s^2), \quad for\ s=1,\ldots,S \tag{1}$$

Applying the principle of convolution, the portfolio composed of these S strategies, characterized by a vector of weightings $\{\omega_s\}$, has excess returns *r* that follow a distribution

$$r \sim N\left(\sum_{s=1}^{S}\omega_s\mu_s, \sum_{s=1}^{S}\left(\omega_s^2\sigma_s^2 + 2\sum_{t=s+1}^{S}\omega_s\omega_t\rho_{s,t}\right)\right) \tag{2}$$

The SR for such portfolio can then be computed as

$$SR = \frac{\sum_{s=1}^{S}\omega_s\mu_s}{\sqrt{\sum_{s=1}^{S}\left(\omega_s^2\sigma_s^2 + 2\sum_{t=s+1}^{S}\omega_s\omega_t\rho_{s,t}\right)}} \tag{3}$$

We would like to investigate the variables that affect the risk-adjusted performance of such portfolio of strategies.

A.2.- BENCHMARK PORTFOLIO

Let's set the benchmark portfolio to be the result of a naïve equal volatility weighting allocation,

$$\omega_s = \frac{1}{S\sigma_s}, \quad for\ s=1,\ldots,S \tag{4}$$

Then, it is immediate to show that the SR of this benchmark portfolio is

$$SR_B = \overline{SR}\sqrt{\frac{S}{1+\frac{2}{S}\sum_{s=1}^{S}\sum_{t=s+1}^{S}\rho_{s,t}}} \tag{5}$$

where $\overline{SR} = \frac{1}{S}\sum_{s=1}^{S}SR_s = \frac{1}{S}\sum_{s=1}^{S}\frac{\mu_s}{\sigma_s}$, the average SR across the strategies. The average correlation across off-diagonal elements is $\bar{\rho} = \frac{2\sum_{s=1}^{S}\sum_{t=s+1}^{S}\rho_{s,t}}{S(S-1)}$. We can compute the SR of the benchmark portfolio as

$$SR_B = \overline{SR}\sqrt{\frac{S}{1+(S-1)\bar{\rho}}} \tag{6}$$

A.3.- SENSITIVITY TO PERFORMANCE DEGRADATION

Let's compute the partial derivative of Eq. (6) with respect to $\overline{SR}$ and $\bar{\rho}$

$$\frac{\partial SR_B}{\partial \overline{SR}} = \sqrt{\frac{S}{1+(S-1)\bar{\rho}}} = \frac{SR_B}{\overline{SR}} \tag{7}$$

$$\frac{\partial SR_B}{\partial \bar{\rho}} = -\frac{\overline{SR}}{2}(S-1)\sqrt{S(1+(S-1)\bar{\rho})} \tag{8}$$

$$\frac{\partial SR_B}{\partial \overline{SR}\partial \bar{\rho}} = -\frac{S-1}{2}\sqrt{S(1+(S-1)\bar{\rho})} \tag{9}$$

Therefore, SR_B is a linear function of the average performance degradation, but a decreasing convex function of the average correlation increase.

A.4.- DIVERSIFICATION

It is interesting to discuss diversification in the context of this benchmark portfolio because we do not assume a skillful capital allocation process. If the capital allocation process is skillful and $\bar{\rho} < 1$, then the portfolio's Sharpe ratio (SR) will surely beat the benchmark's (SR_B). However, if $\bar{\rho} = 1$, then $SR = SR_B = \overline{SR}$ and the capital allocation process cannot benefit from diversification.

We apply Taylor's expansion on Eq. (6) with respect to *S*, to the first order.

$$\Delta SR_B = \frac{\partial SR_B}{\partial S}\Delta S + \sum_{i=2}^{\infty} \frac{\frac{\partial^i SR_B}{\partial S^i}}{i!}\Delta S^i \approx \frac{\overline{SR}(1-\bar{\rho})}{2\sqrt{S}[(1+(S-1)\bar{\rho})]^{\frac{3}{2}}}\Delta S \tag{10}$$

Only when $\bar{\rho} = 0$, the SR can be expanded without limit by increasing *S*. But otherwise, SR gains become gradually smaller until eventually SR_B converges to the asymptotic limit

$$\lim_{S\to\infty} SR_B = \frac{\overline{SR}}{\sqrt{\bar{\rho}}} \tag{11}$$

A.5.- IMPACT OF CANDIDATE STRATEGIES ON THE BENCHMARK

Eq. (11) tells us that the *maximum* SR for the benchmark portfolio is a function of two variables: The average Sharpe ratio among strategies ($\overline{SR}$) and the average off-diagonal correlation ($\bar{\rho}$). It also shows that we should accept a below-average SR strategy if it adds diversification. Going back to Eq. (6), the value of SR_B after adding a new strategy can be updated as

$$SR_B^* = \frac{(\overline{SR} \cdot S + SR_{S+1})}{\sqrt{(S+1) + S[\bar{\rho}(S-1) + 2\bar{\rho}_{S+1}]}} \tag{12}$$

where

- SR_{S+1} is the SR associated with the candidate strategy.
- $\bar{\rho}_{S+1} = \frac{1}{S}\sum_{s=1}^{S} \rho_{s,S+1}$.
- $\rho_{s,S+1}$ are the pairwise correlations between the candidate strategy and the set of S approved strategies.

A.6.- INDIFFERENCE CURVE AND STRATEGY APPROVAL

From Eq. (12), we can isolate an indifference curve for preserving the benchmark's SR, i.e. that imposes the condition $SR_B^* = SR_B$.

$$\bar{\rho}_{S+1} = \frac{1}{2}\left[\frac{(\overline{SR} \cdot S + SR_{S+1})^2}{S \cdot {SR_B}^2} - \frac{S+1}{S} - \bar{\rho}(S-1)\right] \tag{13}$$

This in turn leads to

$$\frac{\partial \bar{\rho}_{S+1}}{\partial SR_{S+1}} = \frac{\overline{SR} \cdot S + SR_{S+1}}{S \cdot {SR_B}^2} \tag{14}$$

And inserting Eq. (6) we derive the equilibrium condition,

$$\frac{\partial \bar{\rho}_{S+1}}{\partial SR_{S+1}} = \frac{(\overline{SR} \cdot S + SR_{S+1})(1 + (S-1)\bar{\rho})}{S^2 \cdot \overline{SR}^2} \tag{15}$$

DISCLAIMER

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