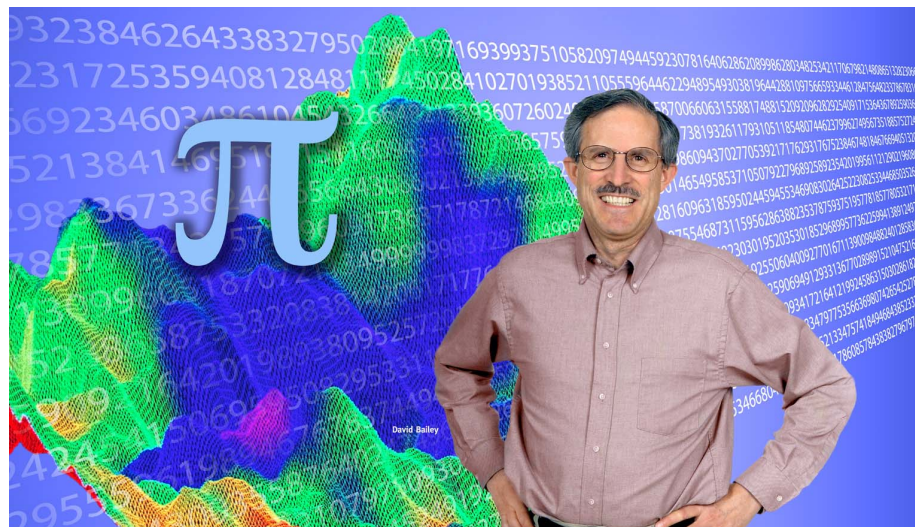# A Quartically Convergent Square Root Algorithm: An Exercise in Forensic Paleo-Mathematics

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David Bailey

A quartically convergent algorithm for Pi: Jon and Peter Borwein's first "big" result

In 1985, Jonathan and Peter Borwein published a "quartically convergent" algorithm for π. "Quartically convergent" means that each iteration approximately **quadruples** the number of correct digits (provided all iterations are performed with full precision):

Set $a_0 = 6$ - sqrt[2], and $y_0 = $ sqrt[2] - 1. Then iterate:

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}}$$

$$a_{k+1} = a_k(1 + y_{k+1})^4 - 2^{2k+3} y_{k+1}(1 + y_{k+1} + y_{k+1}^2)$$

Then a_k, converge quartically to $1/\pi$. This algorithm, together with the Salamin-Brent scheme, has been employed in numerous computations of π.

Both this and the Salamin-Brent scheme are based on the arithmetic-geometric mean and some ideas due to Gauss, but evidently he (nor anyone else until 1976) ever saw the connection to computation. Perhaps no one in the pre-computer age was accustomed to an "iterative" algorithm?

Ref: J. M. Borwein and P. B. Borwein, Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity}, John Wiley, New York, 1987.

A quartically convergent algorithm for square roots

I have found a quartically convergent algorithm for square roots in a little-known manuscript:

♦ To compute the square root of q, let x_0 be the initial approximation. Then iterate:

$$a_n = \frac{q - x_n^2}{2x_n}$$

$$x_{n+1} = x_n + a_n - \frac{a_n^2}{2\left(x_n + a_n\right)}$$

Then x_n converge quartically to sqrt[q]. For example, for q = 336009, x_0 = 579:

Iteration	Value	Relative error
0	579.000000000000000000000000000000. . .	1.143×10^{-3}
1	579.662833033259038411439488233421. . .	2.142×10^{-13}
2	579.662833033134874975589542464552. . .	2.631×10^{-52}
3	579.662833033134874975589542464552. . .	5.993×10^{-208}
4	579.662833033134874975589542464552. . .	1.612×10^{-830}
5	579.662833033134874975589542464552. . .	8.449×10^{-3321}
6	579.662833033134874975589542464552. . .	6.371×10^{-13282}
7	579.662833033134874975589542464552. . .	2.060×10^{-53126}

"In the case of a number whose square root is to be found, divide [the difference from the square of the approximation] by the approximate root; multiply the denominator of the resulting [fraction] by two; square [the result]; halve it; divide it by [the first approximation]; subtract [from the first approximation]; [the result is] the refined root." [Translation].

Note that successive quantities are (in our notation):

$$a_n = \frac{q - x_n^2}{2x_n}, \quad x_n + a_n, \quad a_n^2, \quad \frac{a_n^2}{2}$$

$$\frac{a_n^2}{2(x_n + a_n)}, \quad x_n + a_n - \frac{a_n^2}{2(x_n + a_n)}$$

Problem: Find an accurate rational approximation to the solution of $3/4\ x^2 + 3/4\ x = 7000$ (arising from analysis of progressions).

Answer: $x = (-3 + \text{sqrt}[336009]) / 6$. To calculate an accurate value for sqrt[336009], start with the approximation 579. Note that $q = 336009 = 579^2 + 768$. Then calculate as follows:

$a_0 = (q - x_0^2) / (2\ x_0) = 768 / 1158$

$x + a_0 = 579 + 768 / 1158$

$a_0^2 / (2\ (x_0 + a_0)) = 294912 / 777307500$

Thus the refined root $= 579 + 515225088 / 777307500 = 450576267588 / 777307500$.

Note: This is 579.66283303325903841… (correct to 13 digits).

Sqrt[336009] = 579.66283303313487498…

The manuscript then performs a calculation to check that the original quadratic equation is satisfied. It obtains, for the left-hand side of the quadratic equation:

50753383762746743271936 / 7250483394675000000

which, after subtracting the correction 21743271936 / [7250483394675000000], gives:

50753383762725000000000 / 7250483394675000000 = 7000 exactly.

The scheme can seen to be equivalent to two consecutive iterations of the Newton-Raphson iteration for the solution of $x^2 - q = 0$:

$$x_{n+1} = x_n + \frac{q - x_n^2}{2x_n}$$

$$x_{n+2} = x_{n+1} + \frac{q - x_{n+1}^2}{2x_{n+1}} = x_n + \frac{q - x_n^2}{2x_n} + \frac{q - \left(x_n + \frac{q - x_n^2}{2x_n}\right)^2}{2\left(x_n + \frac{q - x_n^2}{2x_n}\right)}$$

$$= x_n + \frac{q - x_n^2}{2x_n} + \frac{q - x_n^2 - 2x_n\left(\frac{q - x_n^2}{2x_n}\right) - \left(\frac{q - x_n^2}{2x_n}\right)^2}{2\left(x_n + \frac{q - x_n^2}{2x_n}\right)}$$

$$= x_n + \frac{q - x_n^2}{2x_n} - \frac{\left(\frac{q - x_n^2}{2x_n}\right)^2}{2\left(x_n + \frac{q - x_n^2}{2x_n}\right)}$$
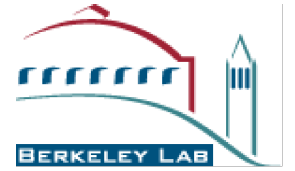
which is the same form as the manuscript iteration. Since two Newton iterations achieve quartic convergence, so does the manuscript iteration.

Contents of the manuscript

- Solutions of systems of linear equations.
- Indeterminate (i.e., Diophantine) equations of the second degree.
- Arithmetic progressions of various types.
- Rational approximations of square roots.
- Additive series.
- Product series.
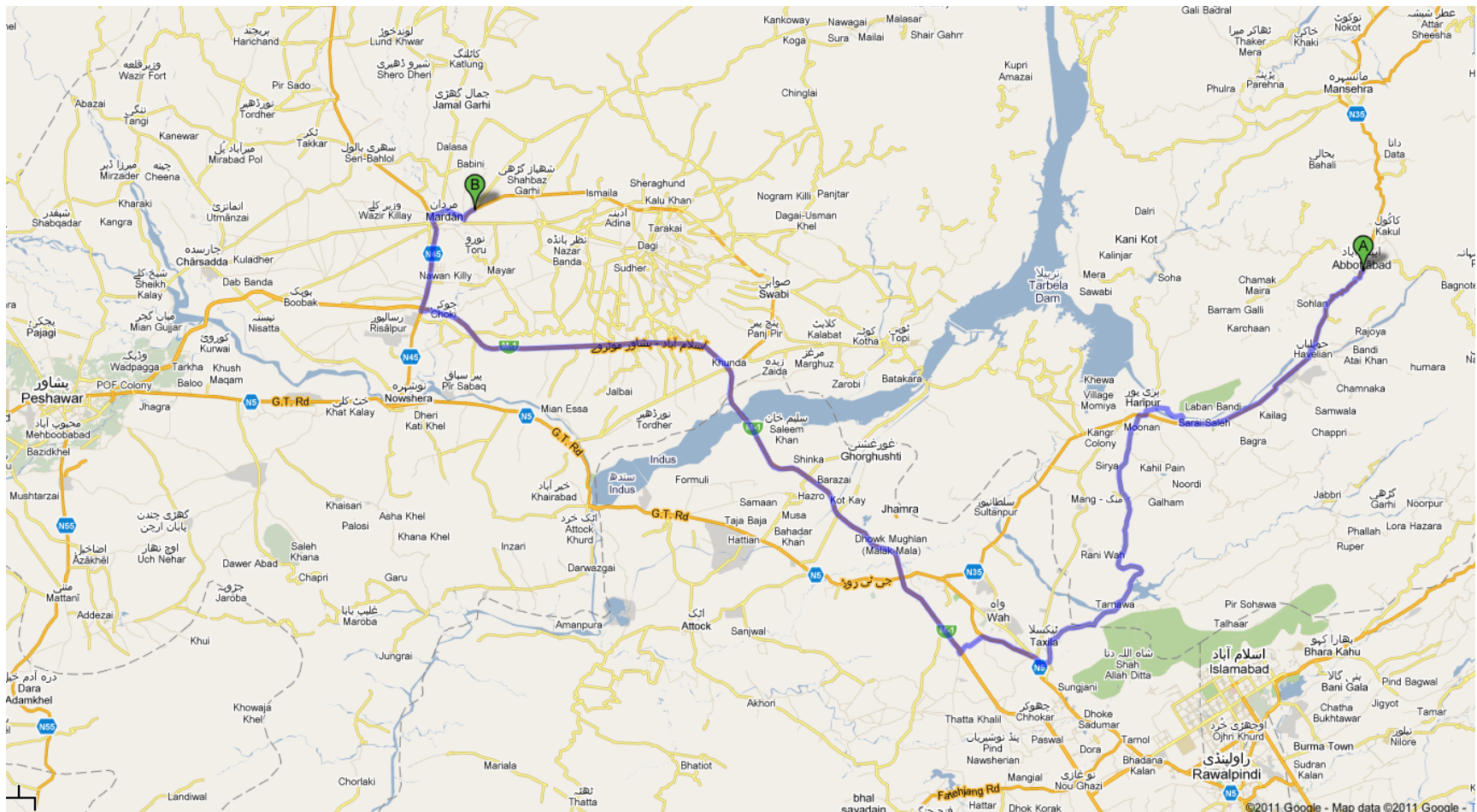- Business applications.
- Geometry.

Title? Author? Date?

Guesses?

Author: An unknown Indian scholar. The manuscript was found in 1881 in Bakhshali (near Peshawar, not far from Afghanistan border), then in India but now in Pakistan. Here is the route to Bakhshali from bin Laden's home in Abbottabad:

The Bakhshali manuscript

Date: The manuscript appears to be a 10-11th century copy of an earlier work. The consensus is that the earlier work was written 0 – 400 CE.

Scholarly opinion:

- Rudolf Hoernle: "3rd or 4th century CE."
    - He notes that the document is written in the "Shloka" style, which was replaced by the "Arya" style prior to 500 CE, and furthermore that it was written in the "Gatha" dialect, which was largely replaced by Sanskrit, at least in secular writings, prior to 300 CE. Also, there are notational features (e.g., + is used for negative) that do not appear in later documents.
- B. Datta: "towards the beginning of the Christian era."
- M. N. Channabasappa: "200-400 CE."
- L. V. Gurjar: "between 2nd century BCE and 2nd century CE."
- Hayashi: "seventh century."
- G. R. Kaye: "12th century."
    - Most present-day scholars agree that Kaye's analysis is flawed.
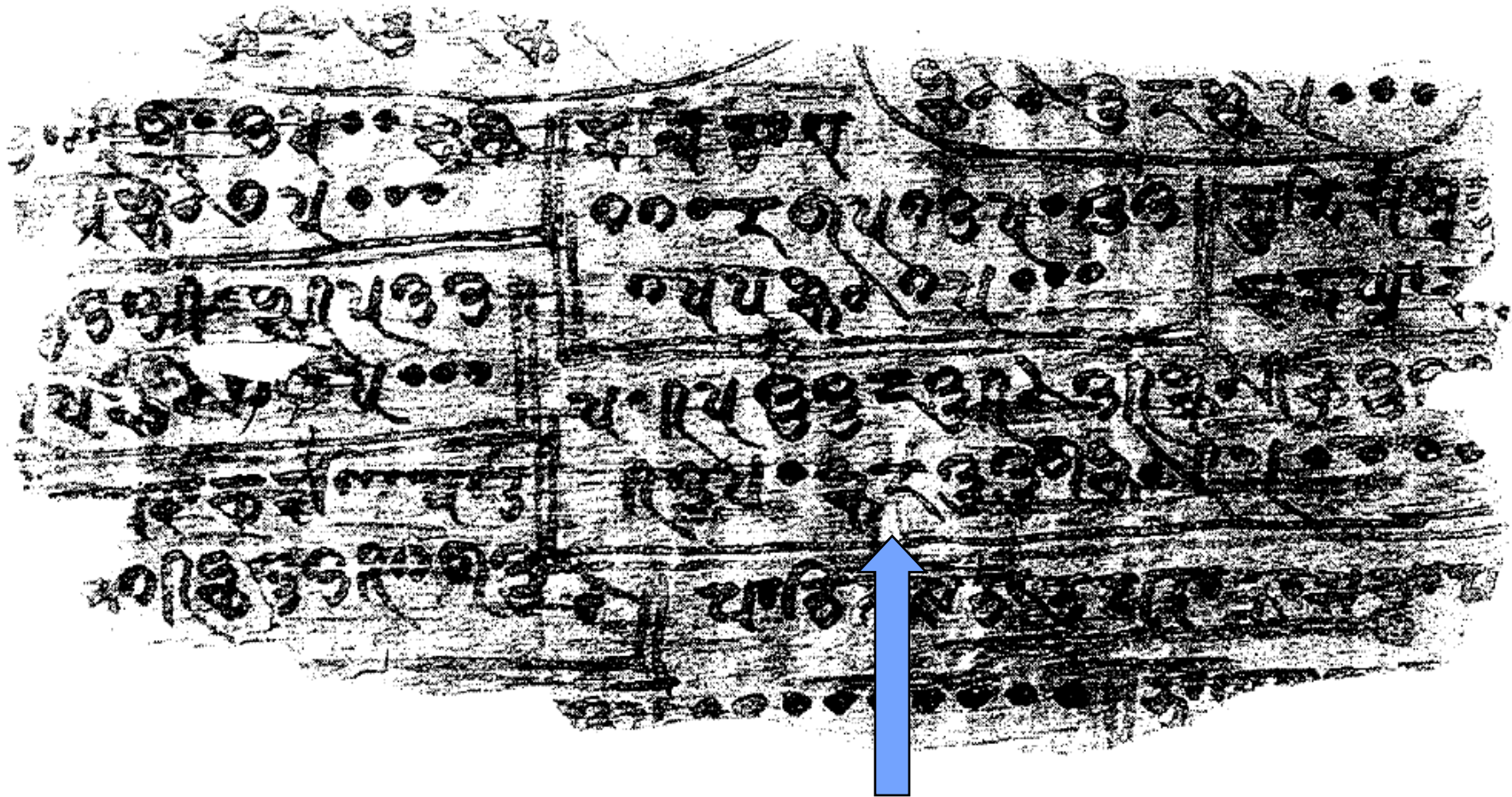
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Even earlier square roots?
Even earlier positional decimal arithmetic?

♦ Datta notes that as early as 300-400 BCE, a scholar wrote, after erroneously assuming that π = sqrt[10], that the "circumference" of a circle of diameter 100,000 yojana (a unit of distance, approx. 11.5 km) is (approximately):

316227 yojana + 3 gavyuti + 128 dhanu + 13 ½ angula, "and a little over."

♦ According to several historical sources, these distance units are:

4 gavyuti = 1 yojana; 2000 dhanu = 1 gavyuti; 96 angula = 1 dhanu.

♦ Thus the "circumference" = 316227.766017578125 yojana.

100000 x sqrt[10] = 316227.766016837933…

This is 12-digit agreement! This has received only scant attention in existing literature.

♦ Did Indian scholars know numerical square root algorithms back then?!

♦ Did Indian scholars know positional decimal arithmetic back then?!

What algorithm was used to compute sqrt[10] to this accuracy?

Note that 13 ½ angula, "and a little over" suggests 13 5/8. We can then compare as follows:

- Manuscript value, with 13 1/2   316227.76601757812500…
- Manuscript value, except with 13 5/8   316227.76601774088541…
- One Heron iteration (316227)   316227.76601776571892…
- One Bakhshali iteration (316000)   316227.76601684858667...
- One Bakhshali iteration (316227)   316227.76601683793319...
- Exact value of 100000 x sqrt[10]   316227.76601683793319…

From these results, it appears that one Newton-Raphson-Heron iteration was performed, with 316227 as the starting value.

Even this calculation would require substantial numerical computation.

Another ancient Indian source:
The Lokavibhaga manuscript

The earliest document that exhibits familiarity with positional decimal arithmetic, and which at the same time can be accurately and unambiguously dated, is the Indian cosmological work *Lokavibhaga* ("Parts of the Universe").

Here, for example, we find numerous huge numbers, such as 14236713, 13107200000 and 70500000000000000, as well as many detailed calculations, e.g., (14230249 - 355684) / 212 = 65446 13/212.

Near the end of the *Lokavibhaga*, the author provides detailed astronomical data that enable modern scholars to confirm, in two independent ways, that this text was written on **25 August 458 CE** (Julian calendar). The text also mentions that it was written in the 22nd year of the reign of Simhavarman, which corresponds to 458 CE.

As French historian Georges Ifrah points out, this information not only allows us to date the document with precision, but also proves its authenticity.

Aryabhata's mathematical treatise

In 510 CE, the Indian mathematician Aryabhata presented schemes not only for various arithmetic operations, but also for square roots and cube roots. Additionally, Aryabhata gave a decimal value of π = 3.1416 (approx.).

A statue of Aryabhata, on display at the Inter-University Centre for Astronomy and Astrophysics (IUCAA) in Pune, India, is shown here:

4 5	4 6	8 0	4 9	6		$\lfloor\sqrt{45}\rfloor = 6$
3 6						$6^2 = 36$
9 4				6 7		$\lfloor 94/(2 \cdot 6)\rfloor = 7$
8 4						$7 \cdot (2 \cdot 6) = 84$
1 0 6						
4 9						$7^2 = 49$
5 7 8				6 7 4		$\lfloor 578/(2 \cdot 67)\rfloor = 4$
5 3 6						$4 \cdot (2 \cdot 67) = 536$
4 2 0						
1 6						$4^2 = 16$
4 0 4 4				6 7 4 3		$\lfloor 4044/(2 \cdot 674)\rfloor = 3$
4 0 4 4						$3 \cdot (2 \cdot 674) = 4044$
0 9						
9						$3^2 = 9$
0						Finished; result = 6743

Aryabhata's "digital" cube root algorithm

								Quotient			Computation
4	5	4	9	9	2	9	3	3			$\lfloor\sqrt[3]{45}\rfloor = 3$
2	7										$3^3 = 27$
1	8	4						3	5		$\lfloor 184/(3\cdot 3^2)\rfloor = 6$ (too high, so take 5)
1	3	5									$5\cdot(3\cdot 3^2) = 135$
	4	9	9								
	2	2	5								$3\cdot 5^2\cdot 3 = 225$
	2	7	4	9							
		1	2	5							$5^3 = 125$
	2	6	2	4	2			3	5	7	$\lfloor 26242/(3\cdot 35^2)\rfloor = 7$
	2	5	7	2	5						$7\cdot(3\cdot 35^2) = 25725$
			5	1	7	9					
			5	1	4	5					$3\cdot 7^2\cdot 35 = 5145$
				3	4	3					
				3	4	3					$7^3 = 343$
						0					Finished; result $= 357$

The significance of positional decimal arithmetic with zero

Pierre-Simon Laplace explained:

♦ *It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to all computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of this achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity.*

French historian Georges Ifrah describes the significance in these terms:

♦ *Now that we can stand back from the story, the birth of our modern number-system seems a colossal event in the history of humanity, as momentous as the mastery of fire, the development of agriculture, or the invention of writing, of the wheel, or of the steam engine.*

The difficulty of pre-decimal arithmetic

Tobias Dantzig (father of George Dantzig, the inventor of linear programming):

♦ *Computations which a child can now perform required then the services of a specialist, and what is now only a matter of a few minutes [by hand] meant in the twelfth century days of elaborate work.*

In the 1500s (before widespread adoption of decimal arithmetic in commerce), a wealthy German merchant, consulting a scholar regarding which European university offered the best education for his son, was told the following:

♦ *If you only want him to be able to cope with addition and subtraction, then any French or German university will do. But if you are intent on your son going on to multiplication and division--assuming that he has sufficient gifts--then you will have to send him to Italy.*

The scientist-Pope Sylvester II

- Gerbert of Aurillac traveled to Spain to learn about Indian arithmetic system first-hand from Arab scholars, then was the first Christian to teach mathematics using decimal arithmetic, all prior to his brief reign as Pope Sylvester II (999-1002 CE).

- Little progress was made at the time, in part because of clerics who rumored that Sylvester II had sold his soul to Lucifer during his travels to Islamic Spain.

- These accusations persisted until 1648, when papal authorities who reopened his tomb reported that Sylvester's body had not been dismembered in penance for Satanic practices.

- Sylvester's reign was a turbulent time, and he died after a short reign. It is worth speculating how history would have been different had this remarkable scientist-Pope lived longer.
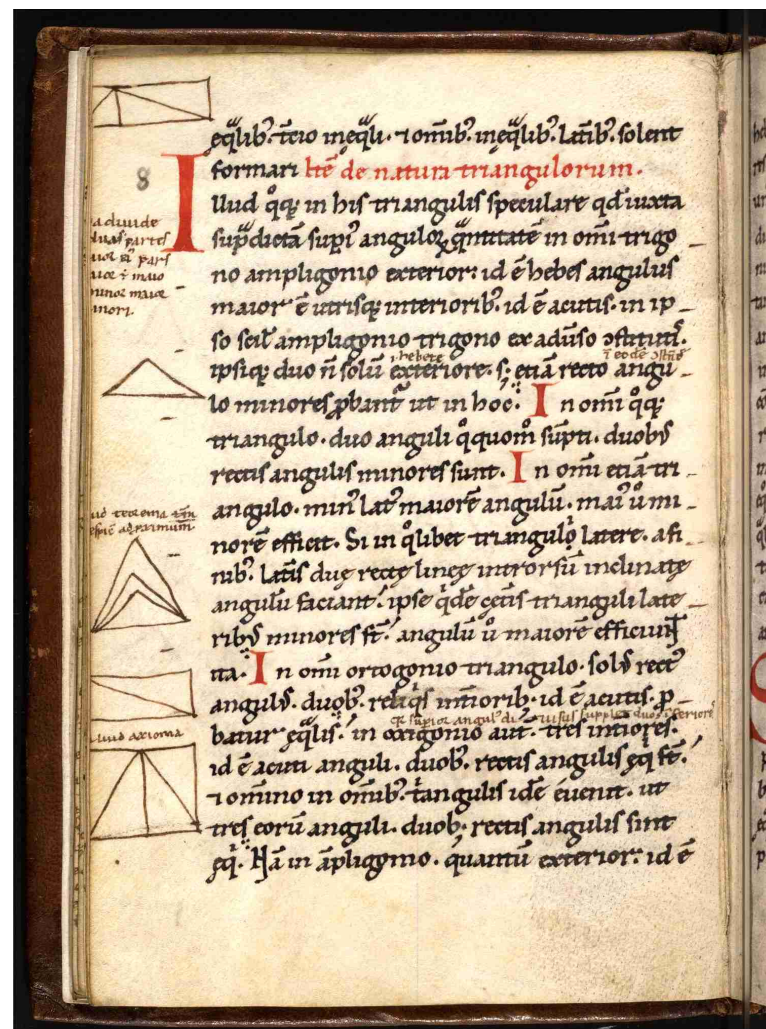
Excerpt from Gerbert's geometry treatise.

Kicking and screaming

- In 1202 CE, Leonardo of Pisa, also known as Fibonacci, reintroduced the Indian system into Europe with his book *Liber Abaci*.

- However, usage of the system remained limited for many years, in part because the scheme was considered "diabolical," due in part to the mistaken impression that it originated in the Arab world (in spite of Fibonacci's clear descriptions of the "nine Indian figures" plus zero).

- Our modern English word "cipher" or "cypher," which is derived from the Arabic zephirum for zero, and which alternately means "zero" or "secret code" in modern usage, is likely a linguistic memory of the time when using decimal arithmetic was deemed evidence of involvement in the occult.

- After much kicking and screaming, Europe finally adopted decimal arithmetic for mathematical and scientific work (1300s and 1400s), but not universally for commerce until nearly 1800.

In other words, positional decimal arithmetic was not fully adopted in Europe until at least 1400 years (and possibly 2000 years) transpired after its discovery in India.

The Indian discovery of decimal arithmetic in present-day math history books

- In one popular work on the history of mathematics, although the author describes Arab and Chinese mathematics in significant detail, he mentions the discovery of positional decimal arithmetic in India only in one two-sentence passage.

- Another popular history of mathematics mentions the discovery of the "Hindu-Arabic Numeral System," but says only that

  - *Positional value and a zero must have been introduced in India sometime before A.D. 800, because the Persian mathematician al-Khowarizmi describes such a completed Hindu system in a book of A.D. 825.*

- A third historical work briefly mentions this discovery, but cites a 662 CE Indian manuscript as the earliest known source.

- A fourth reference states that the combination of decimal and positional arithmetic "appears in China and then in India."

None of these authors devotes more than a few sentences to the subject, and, more importantly, none suggests that this discovery is regarded as particularly significant.

Eurocentric math versus experimental math

Math historian G. G. Joseph writes [*The Crest of the Peakcock: Non-European Roots of Mathematics*, Princeton University Press, 2011, pg. xiii]:

A Eurocentric approach to the history of mathematics is intimately connected with the dominant view of mathematics, both as a sociohistorical practice and as an intellectual activity. Despite evidence to the contrary, a number of earlier histories viewed mathematics as a deductive system, ideally proceeding from axiomatic foundations and revealing, by the "necessary" unfolding of its pure abstract forms, the eternal/ universal laws of the "mind."

The concept of mathematics found outside the Graeco-European praxis was very different. The aim was not to build an imposing edifice on a few self-evident axioms but to validate a result by any suitable method. Some of the most impressive work in Indian and Chinese mathematics …, such as the summations of mathematical series, or the use of Pascal's triangle in solving higher-order numerical equations or the derivations of infinite series, or "proofs" of the so-called Pythagorean theorem, involve computations and visual demonstrations that were not formulated with reference to any formal deductive system.

References

1. David H. Bailey and Jonathan M. Borwein, "The Greatest Mathematical Discovery?," available at http://crd.lbl.gov/~dhbailey/dhbpapers/decimal.pdf
2. David H. Bailey and Jonathan M. Borwein, "Ancient Indian Square Roots: An Exercise in Forensic Paleo-Mathematics," available at http://crd.lbl.gov/~dhbailey/dhbpapers/india-sqrt.pdf
3. Numerous source documents relevant to this topic are available in a protected directory: http://www.dhbailey.com/india-math
   - Virtually all of this material is out-of-print or otherwise very hard to obtain. In one case, the book is available only from five libraries worldwide. Some articles are from obscure and/or defunct Indian math journals. Some obscure (and aging) books have been scanned in their entirety.