A "Hot-Spot" Proof of Normality for the Alpha Constants

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In [1] Richard Crandall and I establish p-normality (i.e., normality base p) for the class of constants

$$\alpha_{p,q} \quad = \quad \sum_{k=1}^{\infty} \frac{1}{q^k p^{q^k}}$$

where p and q are co-prime. The proof given in [1] is somewhat difficult and relies on several not-well-known results, including one by Korobov on the properties of certain pseudo-random sequences. In this note I show that normality can be established much more easily, as a consequence of what may be termed the "weak hot spot" theorem (in the following, $\{\cdot\}$ denotes fractional part):

"Weak Hot Spot" Theorem. The real constant α is b-normal base if and only if there exists a constant C such that for every subinterval $[c, d) \subset [0, 1)$,

$$\limsup_{n \geq 1} \frac{\#_{0 \leq j < n}(\{b^j \alpha\} \in [c, d))}{n} \quad \leq \quad C(d - c).$$

In other words, normal numbers have no "hot spot" intervals, and conversely a non-normal number must have hot spot intervals — there must be digit strings that appear, say, one billion times more often than the frequency they would appear if the number were normal. The weak hot spot theorem is proved in [3, pg. 77].

Here is how the weak hot spot theorem can be used to establish normality for the α constants studied in [1]. In this note I will use $\alpha = \alpha_{2,3}$, namely

$$\alpha \quad = \quad \sum_{k=1}^{\infty} \frac{1}{3^k 2^{3^k}},$$

but the proof is very similar for other $\alpha_{p,q}$ constants from [1].

Theorem. α is normal base 2.

Proof: As above, the notation $\{\cdot\}$ will denote fractional part. First note that the successive shifted binary fractions of α can be written as

$$\{2^n \alpha\} = \left\{ \sum_{m=1}^{\lfloor \log_3 n \rfloor} \frac{2^{n-3^m} \bmod 3^m}{3^m} \right\} + \sum_{m=\lfloor \log_3 n \rfloor + 1}^{\infty} \frac{2^{n-3^m}}{3^m}.$$

As in [1], note that the first term of this expression can be generated by the recursion $x_0 = 0$, and, for $n \geq 1$, $x_n = \{2x_{n-1} + r_n\}$, where $r_n = 1/n$ if $n = 3^k$ for some k, and zero otherwise. Observe that the x sequence has the pattern

$$\begin{array}{l}
0,\ 0,\ 0,\\
\frac{1}{3},\ \frac{2}{3},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{3},\ \frac{2}{3},\\
\frac{4}{9},\ \frac{8}{9},\ \frac{7}{9},\ \frac{5}{9},\ \frac{1}{9},\ \frac{2}{9},\quad \text{(repeated 3 times)},\\
\frac{13}{27},\ \frac{26}{27},\ \frac{25}{27},\ \frac{23}{27},\ \frac{19}{27},\ \frac{11}{27},\ \frac{22}{27},\ \frac{17}{27},\ \frac{7}{27},\ \frac{14}{27},\ \frac{1}{27},\ \frac{2}{27},\ \frac{4}{27},\ \frac{8}{27},\ \frac{16}{27},\ \frac{5}{27},\\
\frac{10}{27},\ \frac{20}{27},\quad \text{(repeated 3 times), etc.}
\end{array}$$

and so forth. It is proven in [1] that indeed this sequence has the pattern evident here: it is a concatenation of triply repeated segments, where each individual segment consists of fractions with numerators, at stage p, that range over all integers relatively prime to the denominator 3^p. We omit this proof here. From this pattern it follows that if $n < 3^{p+1}$ then z_n is a multiple of $1/3^p$. Also, it follows by inspection of (1), that

$$|x_n - \{2^n\alpha\}| \;=\; \left|\sum_{k=n+1}^{\infty} 2^{n-k} r_k\right| \;<\; \frac{1}{2n}. \tag{1}$$

Suppose we are given some half-open interval $[c, d)$. Observe, in view of (1), that if $\{2^j\alpha\} \in [c, d)$, then $x_j \in [c - 1/(2j), d + 1/(2j))$. Let n be any integer greater than $1/(d-c)^2$, and let 3^p denote the largest power of 3 less than or equal to n, so that $3^p \le n < 3^{p+1}$. Let $m = \lfloor 1/(d-c) \rfloor + 1$. Now note that for $j \ge m$, we have $[c - 1/(2j), d + 1/(2j)) \subset [c - (d-c)/2, d + (d-c)/2)$. Since the length of this latter interval is $2(d-c)$, the number of multiples of $1/3^p$ that it contains is either $\lfloor 2 \cdot 3^p(d-c) \rfloor$ or $\lfloor 2 \cdot 3^p(d-c) \rfloor + 1$. Thus there can be at most three times this many j's less than n for which $x_j \in [c - (d-c)/2, d + (d-c)/2)$. Therefore we can write

$$\begin{aligned}
\frac{\#_{0\le j<n}(\{2^j\alpha\} \in [c, d))}{n(d-c)} &\le \frac{m + \#_{m\le j<n}(x_j \in [c - (d-c)/2, d + (d+c)/2))}{n(d-c)}\\
&\le \frac{m + 3(2 \cdot 3^p(d-c) + 1)}{n(d-c)} < 8.
\end{aligned}$$

We have shown that for all $[d-c)$,

$$\limsup_{n\ge 1} \frac{\#_{0\le j<n}(\{2^j\alpha\} \in [c, d))}{n} \;\le\; 8(d-c).$$

so by the weak hot spot theorem, α is 2-normal.

References

[1] David H. Bailey and Richard E. Crandall, "Random Generators and Normal Numbers," *Experimental Mathematics*, vol. 11 (2004), no. 4, pg 527–546, available at `http://crd.lbl.gov/~dhbailey/dhbpapers/bcnormal-em.pdf`.

[2] Patrick Billingsley, *Ergodic Theory and Information*, John Wiley, New York, 1965.

[3] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley-Interscience, New York, 1974.